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On the symmetry of simple 14- and 15-hedra

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The symmetry point-group statistics for all combinatorially non-isomorphic simple 14- (339722) and 15-hedra (2406841 in total) are contributed in the paper for the first time. The most symmetrical shapes with 6 to 48 and 6 to 52 automorphism group orders (61 and 36, respectively) are drawn in Schlegel projections and characterized by facet symbols and symmetry point groups.

1. Introduction

The numbers of combinatorial types of simple 4- to 15-hedra with given combinatorial automorphism group orders were summarized in a recent paper by Engel (2003). In a series of papers (Voytekhovsky, 2001*a*,*b*; Voytekhovsky & Stepenshchikov, 2002*a*,*b*, 2003), we determined the symmetry point groups of all 4- to 11- and simple 12- and 13-hedra that are isomorphous to the related automorphism groups. The symmetry point-group statistics of simple 14-hedra were announced in Stepenshchikov & Voytekhovsky (2002) for the first

a.g.o.	s.p.g.	9-hedra								10-hedra									
		7	8	9	10	11	12	13	14	7	8	9	10	11	12	13	14	15	16
1	1	2	48	237	533	662	449	164	16		44	533	2401	5790	8331	7491	4052	1235	137
2	m	4	17	48	71	87	74	46	18		19	71	145	273	347	341	275	143	51
	2		5	8	22	10	25	3	7	1	8	22	56	58	109	55	87	13	18
	$\bar{1}$												1		3		3		
3	3					1							3			4			1
4	mm2	2	1		7	5	10	4	5	1	2	7	14	13	17	11	19	13	11
	2/m										1		4		6		4		2
	222												1		3		1		
	$\bar{4}$														1				
	4																		
6	3m		2			2			1	2			9			14			6
	32																		
	$\bar{6}$																		
8	4mm			2				2							2				1
	$\bar{4}2m$										1				1				1
	mmm																1		2
10	5m																		
12	$\bar{6}m2$		1			1			2										
	$\bar{3}m$																		
	6mm																		
16	8mm			1															
	$\bar{8}2m$										1								1
	4/mmm														1				
18	9m												1						
20	$\overline{10}m2$									1									
	$\bar{5}m$														1				
	10mm																		
24	$\bar{4}3m$																		1
	$\overline{12}m2$																		
28	$\overline{14}m2$								1										
32	8/mmm																		1
36	$\overline{18}m2$																		
40	10/mmm																		
44	$\overline{22}m2$																		
48	$m\bar{3}m$																		
	12/mmm																		
52	$\overline{26}m2$																		
120	$\bar{3}\bar{5}m$																		
Total		8	74	296	633	768	558	219	50	5	76	633	2635	6134	8822	7916	4442	1404	233
		2606								32300									

Figure 1
Automorphism group order (a.g.o.) and symmetry point group (s.p.g.) statistics of 9- to 11- and simple 12- to 15-hedra.

time. Here, we discuss these data in more detail and contribute the symmetry point-group statistics of simple 15-hedra with Schlegel diagrams of the most symmetrical simple 14- and 15-hedra for the first time.

2. Generation of polyhedra

As the simple 13-hedra were already found, we used them to generate all the simple 14-hedra and, afterwards, simple 15-hedra by the Fedorov recurrence algorithm briefly described in Voytekhovsky (2001*b*). As in all previous cases, we generated the polyhedra as their Schlegel diagrams. This is justified by the theorems of Steinitz (every 3-connected planar graph can be realized as a polyhedron) and Mani (every combinatorial automorphism of a polyhedron is affinely realizable).

The algorithm we used is very close to that described in detail by Engel (2003) with some minor differences. For example, all the simple 12- to 20-hedra with no 3- and 4-gonal facets (118 in total) were found in Voytekhovsky (2001*b*). Among them, one 12-hedron, no 13-hedra, one 14-hedron, one 15-hedron *etc.* with 2-subordination symbols 5_{12}, $5_{12}6_2$ and $5_{12}6_3$, respectively, exist. So what we need to find other simple 14-hedra is to apply the 0-face and 1-face cuts by Engel (*i.e.* α and β procedures by Fedorov) to all simple 13-hedra. In the next step, we apply the same cuts to all simple 14-hedra with the $5_{12}6_2$ polyhedron being already included in the database.

Further, it follows from a well known theorem that there is no polyhedron without 3-, 4- and 5-gonal facets simultaneously. As 3- and 4-gonal facets are generated by 0-face and 1-face cuts, what we need to get the simple (n+1)-hedra with no 3- and 4-gonal facets is to apply the proper 2-face splits (cutting off three consecutive vertices of a single facet, *i.e.* resulting in 5-blades only) of Engel to all simple n-hedra. In accord with what was said above, we do not need this operation to generate the simple 12- to 20-hedra at all.

3. Results and discussion

The automorphism group order statistics of simple 14- and 15-hedra were given by Engel (2003). But only 1 and 3 orders can be *a priori* identified among them as related to 1 and 3 symmetry point groups. So the symmetry statistics for other point groups are suggested in this paper for the first time. They are given in Fig. 1 in comparison with those for all 9- to 11-hedra and simple 12- and 13-hedra discussed in Voytekhovsky & Stepenshchikov (2002*a*,*b*, 2003). The same statistics for all 4- to 8-hedra were published in Voytekhovsky (2001*a*). It follows from Fig. 1 that the automorphism group order statistics of simple 14- and 15-hedra agree with the data by Engel. As in the cases of 4- to 11-hedra and simple 12- and 13-hedra, the shapes of 1, *m*, 2 and *mm*2 symmetry point groups prevail among the simple 14- and 15-hedra with the trivial shapes forming the overwhelming majority.

a.g.o.	s.p.g.	11-hedra											12-hedra	13-hedra	14-hedra	15-hedra
		8	9	10	11	12	13	14	15	16	17	18				
1	1	21	662	5790	24888	63080	102524	110015	78169	35199	9176	970	6756	47030	331796	2382352
2	m	12	87	273	646	1076	1520	1648	1449	1034	492	187	597	1952	6300	20679
	2	1	10	58	72	241	143	375	110	253	24	54	146	425	1237	3315
	$\bar{1}$												4		50	
3	3		1			4			5			1	1	12	12	20
4	mm2	4	5	13	17	25	26	44	25	40	22	27	53	118	211	439
	2/m												10		39	
	222												3		12	
	$\bar{4}$												2		3	
	4														1	
6	3m		2			9			13			5	5	28	24	22
	32					1							1		1	2
	$\bar{6}$												1		2	1
8	4mm														5	
	$\bar{4}2m$												6		5	
	mmm												4		9	
10	5m				2					2			1			
12	$\bar{6}m2$		1			3			2			4	1		4	10
	$\bar{3}m$												2		4	
	6mm														1	
16	8mm															
	$\bar{8}2m$															
	4/mmm														2	
18	9m															
20	$\overline{10}m2$															
	$\bar{5}m$															
	10mm				1											
24	$\bar{4}3m$															
	$\overline{12}m2$														1	
28	$\overline{14}m2$															
32	8/mmm															
36	$\overline{18}m2$											1				
40	10/mmm												1			
44	$\bar{2}2m2$													1		
48	$m\bar{3}m$														2	
	12/mmm														1	
52	$\bar{2}6m2$															1
120	$\bar{3}\bar{5}m$												1			
Total		38	768	6134	25626	64439	104213	112082	79773	36528	9714	1249	7595	49566	339722	2406841
		440564														

Figure 1 (continued)

The most symmetrical 14- and 15-hedra with automorphism group orders not less than 6 are drawn in Schlegel diagrams in Figs. 2 and 3, respectively. A projection is usually made along the main symmetry axis onto the orthogonal facet, if any. It is difficult to draw the projections of 14- and 15-hedra inside the 3- or even 4-gonal facets. Unfortunately, these are the cases for 14-hedra Nos. 16, 24, 34 and most of 15-hedra under discussion. Their main symmetry axes are 3, $\bar{3}$, 4 and $\bar{6}$. In these cases, we drew Schlegel diagrams at one of the biggest *n*-gonal facet. As in our previous papers, we use the facet symbols (in brackets) to lexicographically order the polyhedra. They mean the numbers of 3-, 4-, . . . , *n*-gonal facets in a sequence. The letters *A*, *B*, *C* and *D* stand for the numbers 10, 11, 12 and 13, respectively. Probably, the 2-subordinate symbols are more convenient in the cases when the facet symbols contain many zeros. We use the latter for the sake of uniformity.

14-hedra: [00C2] 1 ($\overline{12}m2$), [0284] 2 (*mmm*), 3 (4/*mmm*), [0446] 4 (*mmm*), [054401] 5 (4*mm*), [0608] 6 ($\bar{4}2m$), 7 ($m\bar{3}m$), [0634001] 8 (3*m*), [0660002] 9 ($\bar{6}m2$), [08024] 10 ($\bar{4}2m$), 11 (*mmm*), [080402] 12 (*mmm*), 13 (4/*mmm*), [08400002] 14 (*mmm*), [090203] 15 ($\bar{6}m2$), [0A0004] 16 ($\bar{4}2m$), [0C00000002] 17 (12/*mmm*), [1337] 18 (3*m*), [13613] 19–20

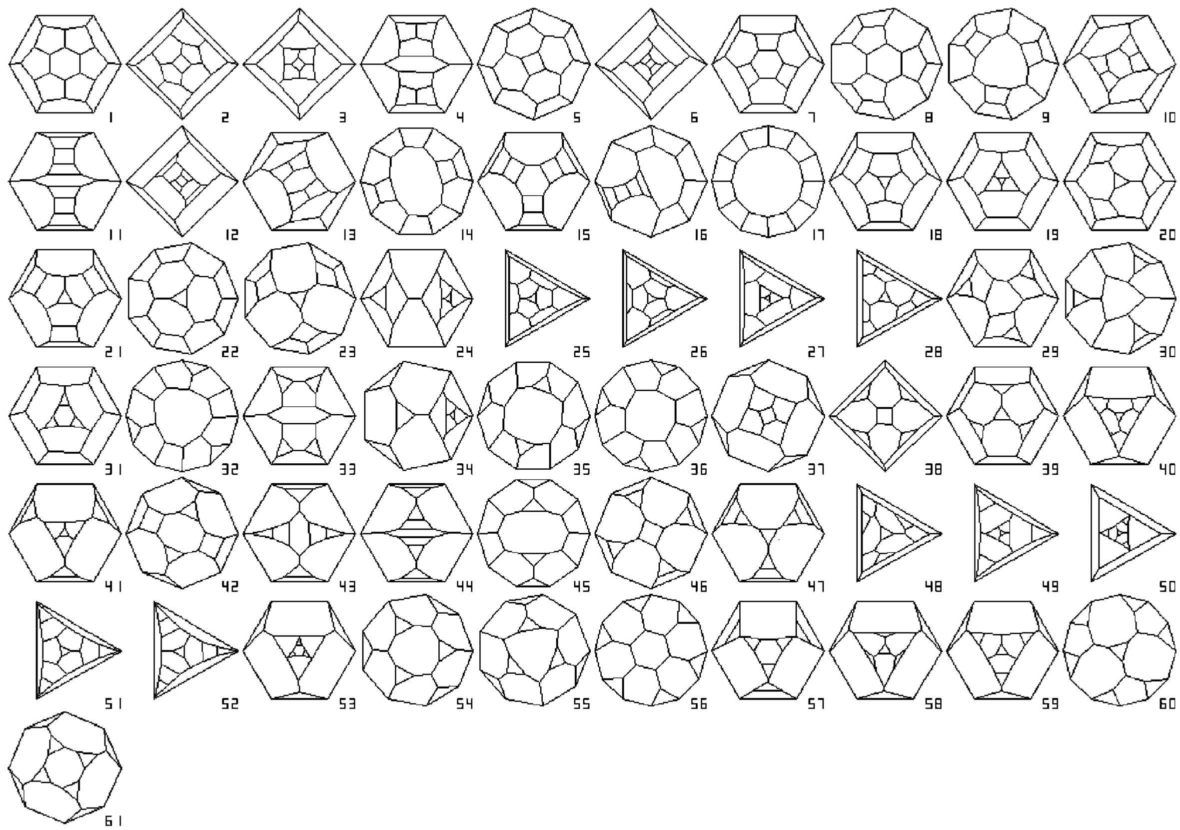

Figure 2
The most symmetrical simple 14-hedra.

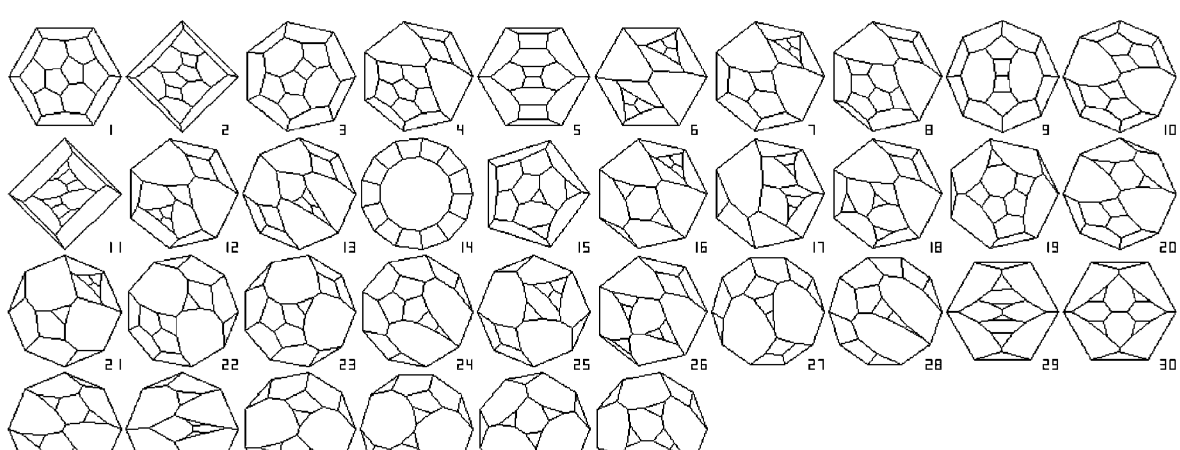

Figure 3
The most symmetrical simple 15-hedra.

(3*m*), [16043] 21 (3*m*), [1630301] 22 (3*m*), [1900031] 23 (3*m*), [2066] 24–25 ($\bar{3}m$), [2309] 26 ($\bar{6}m2$), [23333] 27 (3*m*), [26006] 28 (32), [30623] 29 ($\bar{6}$), [3307001] 30 (3*m*), [333203] 31 (3*m*), [3360001001] 32 (3*m*), [400A] 33 (*mmm*), [40343] 34 (3*m*), [40800002] 35 ($\bar{4}2m$), [4080010001] 36 (4*mm*), [414041] 37 (4*mm*), [42044] 38 (4*mm*), [430403] 39–40 (3*m*), [4331003] 41 (3*m*), [440042] 42 (*mmm*), [440204] 43 ($\bar{4}2m$), 44 (*mmm*), [44040002] 45 (*mmm*), [450005] 46 (4*mm*), [46010003] 47 (3*m*), [50306] 48 ($\bar{6}$), 49 (3*m*), [503303] 50–51 (3*m*), [5303003] 52 (3*m*), [60026] 53 ($\bar{3}m$), [6006002] 54 ($\bar{3}m$), 55 ($\bar{6}m2$), [6007000001] 56 (6*mm*), [63020003] 57 (3*m*), [700133] 58 (3*m*), [7004003] 59 (3*m*), [7030030001] 60 (3*m*), [800006] 61 ($m\bar{3}m$).

15-hedra: [00C3] 1 ($\bar{6}m2$), [0366] 2 (32), [03903] 3–4 (3*m*), [0609] 5–6 ($\bar{6}m2$), [06333] 7–8 (3*m*), [066003] 9–10 ($\bar{6}m2$), [09006] 11 (32), 12 ($\bar{6}m2$), [090303] 13 (3*m*), [0D000000002] 14 ($\bar{2}6m2$), [30633] 15 (3*m*), [33063] 16 (3*m*), [33306] 17 ($\bar{6}$), 18–19 (3*m*), [333303] 20–21 (3*m*), [3360003] 22–24 (3*m*), [360033] 25–26 (3*m*), [36300003] 27–28 (3*m*), [60036] 29–30 ($\bar{6}m2$), [600603] 31 ($\bar{6}m2$), [603033] 32 (3*m*), [6033003] 33 (3*m*), [60600003] 34 ($\bar{6}m2$), [6300303] 35–36 (3*m*).

Engel (2003) reported 1909 different 2-subordination symbols with 4692 different shapes in the maximum series among the simple 14-hedra and 3713 different 2-subordination symbols with 21 203 different shapes in the maximum series among the simple 15-hedra. We also confirm these data.

4. Conclusions

Up to now, all the varieties of 4- to 11- and simple 12- to 15-hedra have been enumerated and characterized by the facet symbols and symmetry point groups. The most symmetrical shapes were drawn in Schlegel projections. The next steps are to generate and characterize all non-simple 12- and simple 16-hedra in the same way.

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References

Engel, P. (2003). *Acta Cryst.* A**59**, 14–17.

Stepenshchikov, D. G. & Voytekhovsky, Y. L. (2002). *Proc. XIII Conf. on Geology and Geoecology (Apatity, Nov. 19–21, 2002)*, Vol. 2: *Mineralogy and Crystallography*, pp. 97–100. Apatity: Kola Science Centre Press. (In Russian.)

Voytekhovsky, Y. L. (2001*a*). *Acta Cryst.* A**57**, 112–113.

Voytekhovsky, Y. L. (2001*b*). *Acta Cryst.* A**57**, 475–477.

Voytekhovsky, Y. L. & Stepenshchikov, D. G. (2002*a*). *Acta Cryst.* A**58**, 404–407.

Voytekhovsky, Y. L. & Stepenshchikov, D. G. (2002*b*). *Acta Cryst.* A**58**, 502–505.

Voytekhovsky, Y. L. & Stepenshchikov, D. G. (2003). *Acta Cryst.* A**59**, 195–198.